physical point of view. Theories of this class make it possible to consider the gravitational field a physical field in the spirit of Faraday–Maxwell and possess all 10 integrals of the motion for a closed system of interacting fields. The effective Riemannian space–time used to describe the motion of matter in theories of this class reflects in a natural way the existence of a physical gravitational field and a single pseudo-Euclidean space–time.

Hence, we again arrive at the necessity of primary study of the possibilities of constructing a gravitational theory realizing the field approach to the description of the gravitational interaction.

12. Conservation Laws for the Gravitational Field and Matter

We shall study the character of conservation laws for all local theories of class (A) without making a specific choice of the Lagrangian density. Proceeding from the basic principles of the field approach, for theories of this class we write the Lagrangian densities of a system consisting of matter and gravitational field in the form

$$L=L_g(\gamma_{ni},\varphi_{ni})+L_M(g_{ni},\varphi_A), \tag{12.1}$$

where γ_{ni} is the metric tensor of pseudo-Euclidean space–time, φ_{ni} is the gravitational field, and φ_A are the remaining fields of matter.

We shall assume with no loss of generality that the metric tensor of Riemannian space–time g_{ni} is a local function depending on the metric tensor of flat space–time, the gravitational field φ_{ni}, and their partial derivatives through second order:

$$g_{ml}=g_{ml}(\gamma_{ni},\partial_s\varphi_{ni},\varphi_{ni},\partial_{sj}\varphi_{ni},\partial_s\gamma_{ni},\partial_{sj}\gamma_{ni},\partial_s\gamma^{nl},\partial_{sj}\gamma^{nl},\gamma^{ni}), \tag{12.2}$$

where we have used the notation

$$\partial_{ns}\varphi=\frac{\partial^2\varphi}{\partial x^n\partial x^s}.$$

We shall assume that the Lagrangian density of matter L_M depends only on the fields φ_A, their partial derivatives of first order, and the metric tensor g_{ni}. It is easy to see that in this case the Lagrangian density of matter contains partial derivatives of the gravitational field through second order.

We shall assume that the Lagrangian density of the gravitational field depends on the metric tensor γ_{ni}, the gravitational field φ_{ni}, and their partial derivatives through third order.

To obtain conservation laws we use the covariant method of infinitesimally small displacmenets. Since the action J is a scalar, for an arbitrary small coordinate transformation (2.12) the variations of the action of matter δJ_M and of the gravitational field δJ_g will be equal to zero.

Since the Lagrangian density of matter contains both covariant and contravariant components of the metric tensor of Riemannian space–time, we vary the Lagrangian density with respect to them as if they were independent and then use the relations between their variations

$$\delta g^{nm}=-g^{ni}g^{ml}\delta g_{il}.$$

We proceed in an altogether similar way in the variation with respect to the components γ_{ni} and γ^{ni} of the metric tensor of flat space–time.

We write the variation of the action of matter under transformation (2.12) in the form

$$\delta J_M=\int d^4x\left\{\frac{\Delta L_M}{\Delta g_{ni}}\,\delta_L g_{ni}+\frac{\delta L_M}{\delta\varphi_A}\,\delta_L\varphi_A+\mathrm{Div}\right\}, \tag{12.3}$$

where Div denotes divergence terms whose consideration is inconsequential for our purpose.

Introducing the notation

$$t_M^{nm}=-2\,\frac{\Delta L_M}{\Delta\gamma_{nm}}=-2\left(\frac{\delta L_M}{\delta\gamma_{mn}}-\gamma^{ns}\gamma^{ml}\,\frac{\delta L_M}{\delta\gamma^{sl}}\right)$$
$$t_{Mn}^{m}=\gamma_{ns}t_M^{ms} \tag{12.4}$$

for the density of the energy—momentum tensor of matter in flat space—time, we can write the variation of the action integral δJ_M under the coordinate transformation (2.12) in another form equivalent to expression (12.3):

$$\delta J_M=\int d^4x\left\{\frac{\delta L_M}{\delta\varphi_{nm}}\delta_L\varphi_{nm}+\frac{\Delta L_M}{\Delta\gamma_{nm}}\delta_L\gamma_{nm}+\frac{\delta L_M}{\delta\varphi_A}\delta_L\varphi_A+\mathrm{Div}\right\}. \tag{12.5}$$

The variations $\delta_L\gamma_{nm}$, $\delta_L\varphi_{ln}$, $\delta_L\varphi_A$, and $\delta_L g_{nm}$ under coordinate transformations (2.12) have the form

$$\begin{gathered}\delta_L\gamma_{nm}=-\gamma_{ni}D_m\xi^i-\gamma_{mi}D_n\xi^i;\\ \delta_L\varphi_A=-\xi^iD_i\varphi_A+F^{B;n}_{A;i}\varphi_BD_n\xi^i;\\ \delta_L\varphi_{nm}=-\varphi_{ni}D_m\xi^i-\varphi_{mi}D_n\xi^i-\xi^iD_i\varphi_{nm};\\ \delta_Lg_{nm}=-g_{ni}D_m\xi^i-g_{mi}D_n\xi^i-\xi^iD_ig_{nm}.\end{gathered} \tag{12.6}$$

Considering these equalities, the variation of the action integral of matter (12.5) can be written in the form

$$\delta J_M=0=\int d^4x\left\{\xi^l\left[2D_n\left(\frac{\delta L_M}{\delta\varphi_{nm}}\varphi_{ml}\right)-D_nt^n_{Ml}-\frac{\delta L_M}{\delta\varphi_{nm}}D_l\varphi_{nm}-D_n\left(\frac{\delta L_M}{\delta\varphi_A}F^{B;n}_{A;l}\varphi_B\right)-\frac{\delta L_M}{\delta\varphi_A}D_l\varphi_A\right]+\mathrm{Div}\right\}. \tag{12.7}$$

Since the displacement vector ξ^l in expression (12.7) is arbitrary, from this we obtain the following identity (a strong conservation law):

$$D_lt^l_{Mn}-2D_l\left(\frac{\delta L_M}{\delta\varphi_{ml}}\varphi_{mn}\right)+\frac{\delta L_M}{\delta\varphi_{lm}}D_n\varphi_{ml}+D_l+\left(\frac{\delta L_M}{\delta\varphi_A}F^{B;l}_{A;n}\varphi_B\right)+\frac{\delta L_M}{\delta\varphi_A}D_n\varphi_A=0. \tag{12.8}$$

We obtain another important identity if we substitute relations (12.6) in the expression (12.3):

$$D_l(g_{nm}T^{lm})-\frac{1}{2}T^{ml}D_ng_{ml}=-D_l\left(\frac{\delta L_M}{\delta\varphi_A}F^{B;l}_{A;n}\varphi_B\right)-\frac{\delta L_M}{\delta\varphi_A}D_n\varphi_A. \tag{12.9}$$

We now express the covariant derivatives on the left side of identity (12.9) in terms of the partial derivatives and connections of flat space—time γ^i_{nl}. Noting that T^{ni} is a tensor density of weight l, we obtain

$$\partial_l(g_{nm}T^{ml})-\frac{1}{2}T^{ml}\partial_ng_{ml}=-D_l\left(\frac{\delta L_M}{\delta\varphi_A}F^{B;l}_{A;n}\varphi_B\right)-\frac{\delta L_M}{\delta\varphi_A}D_n\varphi_A.$$

Now the left side of this expression is the covariant divergence in Riemannian space—time of the density of the energy—momentum tensor of matter T^i_n:

$$\partial_l(g_{nm}T^{ml})-\frac{1}{2}T^{ml}\partial_ng_{ml}=\nabla_lT_n{}^l=g_{nm}\nabla_lT^{ml}.$$

Therefore, relation (12.9) assumes the form

$$g_{ni}\nabla_lT^{li}=-D_i\left(\frac{\delta L_M}{\delta\varphi_A}F^{B;i}_{A;n}\varphi_B\right)-\frac{\delta L_M}{\delta\varphi_A}D_n\varphi_A.$$

Subtracting this equality from expression (12.8), we obtain

$$D_it^i_{Mn}-2D_i\left(\frac{\delta L_M}{\delta\varphi_{mi}}\varphi_{mn}\right)+\frac{\delta L_M}{\delta\varphi_{ml}}D_n\varphi_{ml}=g_{ni}\nabla_lT^{li}. \tag{12.10}$$

It should be emphasized that this identity holds independently of the equations of motion of matter and the gravitational field, and it is hence a strong conservation law.

In a similar way from the invariance of the action of the gravitational field under the transformation (2.12), we obtain

$$D_it^i_{gn}-2D_i\left(\frac{\delta L_g}{\delta\varphi_{mi}}\varphi_{nm}\right)+\frac{\delta L_g}{\delta\varphi_{ml}}D_n\varphi_{ml}=0. \tag{12.11}$$

For the density of the symmetric energy—momentum tensor of the gravitational field t^i_{gn} we have as usual

$$t^i_{gn} = -2\gamma_{nm} t^{mi}_g = -2\gamma_{nm} \frac{\Delta L_g}{\Delta \gamma_{mi}}. \qquad (12.12)$$

From relations (12.10) and (12.11) it follows that

$$D_i (t^i_{Mn} + t^i_{gn}) - 2D_i \left(\frac{\delta L}{\delta \varphi_{mi}} \varphi_{mn}\right) + \frac{\delta L}{\delta \varphi_{ml}} D_n \varphi_{ml} = \nabla_i T_n{}^i. \qquad (12.13)$$

Under the condition that the equations of the gravitational field are satisfied

$$\frac{\delta L}{\delta \varphi_{nm}} = \frac{\delta L_g}{\delta \varphi_{nm}} + \frac{\delta L_M}{\delta \varphi_{nm}} = 0 \qquad (12.14)$$

expression (12.13) simplifies:

$$D_i [t^i_{Mn} + t^i_{gn}] = g_{ni} \nabla_m T^{mi}. \qquad (12.15)$$

The equality is a manifestation of the principle of geometrization. From it it follows that the covariant divergence in pseudo-Euclidean space—time of the sum of the tensor densities of the energy—momentum of matter and of the gravitational field is transformed into a covariant divergence in Riemannian space—time of the density of the energy—momentum tensor of matter alone. Thus, these are different forms of writing the same expression.

Under the condition that the equations of the motion of matter (2.11) are satisfied, expression (12.8) simplifies to

$$D_i t^i_{Mn} - 2D_i \left(\frac{\delta L_M}{\delta \varphi_{li}} \varphi_{ln}\right) + \frac{\delta L_M}{\delta \varphi_{ml}} D_n \varphi_{ml} = 0, \qquad (12.16)$$

and from relation (12.9) there automatically follows the covariant equation for conservation of the density of the energy—momentum tensor of matter in Riemannian space—time:

$$\nabla_n T^{ni} = \partial_n T^{ni} + \Gamma^i_{ml} T^{ml} = 0. \qquad (12.17)$$

This equation is general for theories with a geometrized Lagrangian density of matter and is not connected with any concrete version of the theory of gravitation.

Further, we see that from relations (12.16) and (12.11) under the condition that the equations of the gravitational field (12.14) are satisfied there follows a covariant conservation law for the density of the total symmetric energy—momentum tensor in pseudo-Euclidean space—time:

$$D_i [t^i_{Mn} + t^i_{gn}] = 0. \qquad (12.18)$$

Thus, on the basis of the Lagrangian formalism we have obtained a conservation law of the energy—momentum of matter and the gravitational field in pseudo-Euclidean space—time. This fundamental law of nature means that in the field theory of gravitation there are no processes (regardless of the erudition of their inventors) which proceed without conservation of energy—momentum. From expression (12.18) it follows also that the gravitational field considered in pseudo-Euclidean space—time behaves like other physical fields. It possesses energy—momentum and contributes to the total energy—momentum tensor of the system.

On the basis of Eq. (12.18) and identity (12.15) we obtain

$$D_i [t^i_{Mn} + t^i_{gn}] = g_{nl} \nabla_i T^{il} = 0.$$

Thus, the conservation law for the density of the total energy—momentum tensor (12.18) and the conservation law in form (12.17) when the equation of the gravitational field (12.14) and the equations of motion of matter (12.11) are satisfied represent simply different forms of writing the same conservation law. The conservation law (12.18) expresses the fact that in pseudo-Euclidean space—time the density of the total energy-momentum tensor consisting of matter and gravitational field is conserved. This law has the usual form of a conservation law. The conservation law (12.17) in Riemannian space—time is not a conservation law in the usual sense, since the density of the energy—momentum tensor of matter need not be conserved:

$$\partial_n T^{ni} \neq 0.$$

As Einstein already indicated [18, p. 492]: "... The presence of the second term on the left side from a physical point of view means that for matter alone the laws of conservation of momentum and energy in their genuine sense are not satisfied; more precisely, they are satisfied only when $g_{\mu\nu}$ are constant, i.e., when the components of intensity of the gravitational field are equal to zero. The second term represents an expression for the momentum and, correspondingly, for the energy which are transmitted to matter from the gravitational field per unit time in unit volume... ."

In this case the second term in (2.17) expresses the energetic action of the gravitational field on matter and shows that matter obtains energy which is "stored," as it were, in the Riemannian geometry. The energy of the gravitational field in this case effectively goes to the creation of the Riemannian geometry. However, from expression (12.17) it is not evident that quantity is conserved.

The lack of conservation laws in the genuine sense is inherent in the entire subclass of gravitational theories with total geometrization and not only Einstein's theory. The Lagrangian density of the gravitational field L_g of theories of this subclass depends on the field φ_{ni} and the metric tensor γ_{ni} only through the metric tensor of Riemannian space–time g_{ni}. Therefore, in theories of this subclass for the density of the symmetric energy–momentum tensor of matter and gravitational field in pseudo-Euclidean space–time we have

$$-\frac{1}{2}t^{ni}=\frac{\Delta L}{\Delta\gamma_{ni}}=\frac{\Delta L_g}{\Delta\gamma_{ni}}+\frac{\Delta L_M}{\Delta\gamma_{ni}}=\frac{\Delta L}{\Delta g_{ml}}\frac{\partial g_{ml}}{\partial\gamma_{ni}}-\partial_s\left[\frac{\Delta L}{\Delta g_{ml}}\frac{\partial g_{ml}}{\partial(\partial_s\gamma_{ni})}\right]+$$

$$+\partial_{sq}\left[\frac{\Delta L}{\Delta g_{ml}}\frac{\partial g_{ml}}{\partial(\partial_{sq}\gamma_{ni})}\right]-\gamma^{is}\gamma^{nj}\left\{\frac{\Delta L}{\Delta g_{ml}}\frac{\partial g_{ml}}{\partial\gamma^{sj}}-\partial_q\left[\frac{\Delta L}{\Delta g_{ml}}\frac{\partial g_{ml}}{\partial(\partial_q\gamma^{sj})}-\partial_r\left(\frac{\Delta L}{\Delta g_{ml}}\frac{\partial g_{ml}}{\partial(\partial_{qr}\gamma^{sj})}\right)\right]\right\}.$$

Since in a geometrized theory the equations of the gravitational field have the form

$$\frac{\Delta L}{\Delta g_{ml}}=\frac{\delta L}{\delta g_{ml}}-g^{ls}g^{mn}\frac{\delta L}{\delta g^{ns}}=0,$$

the density of the symmetric energy–momentum tensor of matter and gravitational field in pseudo-Euclidean space–time vanishes because of the equations of the gravitational field:

$$\frac{\Delta L}{\Delta\gamma_{ni}}=-\frac{1}{2}t^{ni}=0. \tag{12.19}$$

An analogous conclusion regarding the vanishing of the density of the symmetric energy–momentum tensor is also obtained for the free gravitational field. However, the equations of the free gravitational field have solutions for which the curvature tensor R^i_{nlm} is nonzero. Therefore, in theories with total geometrization the vanishing of the density of the energy–momentum tensor of the free gravitational field does not lead to vanishing of the field φ_{ni}, and hence there exists some fictitious field not possessing a density of energy–momentum but leading to the curving of space–time (the formation of the Riemannian geometry).

The subclass of gravitational theories with total geometrization, in principle, do not enable us to introduce the concept of a gravitational field possessing energy–momentum. From this it follows that any path of constructing a theory of gravitation based on flat space–time and proceeding from perceptions of the gravitational field as a physical field with a nonzero energy–momentum tensor in principle cannot lead to Einstein's general theory of relativity. This general conclusion proves the fallacy of the assertions of a number of authors [5, 12] that all such theories unavoidably reduce to Einstein's theory.

We thus arrive at the following conclusions. 1. In local theories of class (A) the gravitational field described in pseudo-Euclidean space–time is a physical field possessing energy–momentum. On the basis of the identity principle, the motion of matter is described in an effective Riemannian space–time created by the energy–momentum of the gravitational field. In this approach geometric description arises on the basis of field-theoretic conceptions of the gravitational field, and conservation laws lie at the basis of it. 2. In the subclass of theories with total geometrization the gravitational field and matter have a single geometry, but the gravitational field loses the properties of a physical field; it does not possess a density of energy–momentum. In this approach there are no field-theoretic conceptions of the gravitational field as a field in the spirit of Faraday–Maxwell.

The general theory of relativity realizes one possibility of constructing a theory. It introduced a field of new type described by a curvature tensor which is not a Faraday–Maxwell field. Therefore, here there are no conservation laws of matter and gravitational field taken together, and this theory does not satisfy the principle of correspondence with the gravitational theory of Newton.

13. Gauge-Invariant Equations of the Gravitational Field

In this section all relations and equations we shall formulate in Cartesian coordinates, although they can, of course, be written in covariant fashion and in an arbitrary curvilinear coordinate system.

We shall consider theories of class (A) with a Lagrangian density of the form (12.1). The equations of the gravitational field and the equations of motion of matter have the form

$$\frac{\delta L_g}{\delta \varphi_{nm}}+\frac{\delta L_M}{\delta \varphi_{nm}}=0, \tag{13.1}$$

$$\frac{\delta L_M}{\delta \varphi_A}=0. \tag{13.2}$$

In the set of theories with Lagrangian density (12.1) there are theories in which the action integral is invariant under gauge transformations:

$$\varphi_{ni}\to\varphi_{ni}+\partial_n a_i+\partial_i a_n, \tag{13.3}$$

where a_n is an arbitrary gauge four-vector. From invariance of the action integral of the free gravitational field under the gauge transformation (13.3) we have

$$\delta J_g=\int\left[-2a_n\partial_i\frac{\delta L_g}{\delta\varphi_{ni}}+\mathrm{Div}\right]d^4x=0.$$

Since the gauge vector a_n is arbitrary, we obtain

$$\partial_i\frac{\delta L_g}{\delta\varphi_{ni}}=0.$$

From this equation and the field equation (13.1) we obtain the conservation equation for the source of the gravitational field

$$\partial_i\frac{\delta L_M}{\delta\varphi_{ni}}=0.$$

As is known [2], in electrodynamics the invariance of the Lagrangian density $L = L_A + L_M$ under gauge transformations of the vector potential $A_i \to A_i + \partial_i f$ imply analogous conservation laws:

$$\partial_i\frac{\delta L_M}{\delta A_i}=0,\quad \partial_i\frac{\delta L_A}{\delta A_i}=0.$$

Since in a gauge theory the source in the field equations is conserved, it is usually assumed that the source in equations of a gauge theory of gravitation is the total energy–momentum tensor of the system of matter plus gravitational field. This leads to the situation that the field equations become nonlinear, and it is usually asserted that the consistent inclusion of such nonlinearities can lead to Einstein's nonlinear theory of gravitation [5, 12, 22].

However, on the one hand, this hypothesis leads primarily to the fact that the gravitational field looses the properties of a carrier of energy–momentum. If the possibility is assumed of identifying the source

$$\frac{\delta L_M}{\delta\varphi_{ni}}=\frac{1}{2}\mathcal{T}^{ni}$$

with the total energy–momentum tensor $t^{ni} = t^{ni}_g + t^{ni}_M$, then this immediately implies that the energy–momentum tensor of the free gravitational field (for $L_M = 0$) is equal to zero. Such a theory does not possess properties characteristic of other physical systems, and we therefore consider it unacceptable. On the other hand, contrary to the assertions of the authors of [5, 12, 22], a path proceeding from concepts of a physical gravitational field with a nonzero energy–momentum tensor does not in principle lead to Einstein's theory, as shown in Sec. 12.